SUMMARY

VIGGO BRUN: *Leibniz' formula for π deduced by a »mapping« of the circular disc.* (English.)

In the eighth part of a circle with radius 1 (figs. 9 and 10 p. 79), we draw the curves C_n whose equations in polar coordinates are $r=\mathrm{tg}^{2n}v$, $0\leqq v\leqq \pi/4$, $n=1,2,3,\ldots$. Then the area enclosed by the axis, the circle and C_1 is $\frac{1}{2}(1-\frac{1}{3})$, the area between C_1 and C_2 is $\frac{1}{2}(\frac{1}{5}-\frac{1}{7})$, the area between C_2 and C_3 is $\frac{1}{2}(\frac{1}{9}-\frac{1}{11})$, and so on. Summing these areas, we get the celebrated formula of Leibniz:

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots.$$

LEIBNIZ' FORMULA FOR π DEDUCED BY A "MAPPING" OF THE CIRCULAR DISC*

VIGGO BRUN

In this journal I have published an article [1] in 1955: "On the problem of partitioning the circle so as to visualize Leibniz' formula for π." I began with quoting an interesting remark by Lionardo da Vinci [4]. In his opinion the art of painting—the art of the eye—is superior to poetry —the art of the ear—because the eye is a much finer organ than the ear. He says: "If you, historians, or poets, or mathematicians had not seen things with your eyes you could not report them in writing."

I had got the idea to my research by reading an article of lord Brouncker [2] from 1668, where he gave a "squaring of the hyperbola, by an infinite series of rational numbers", namely the area A under the hyper-

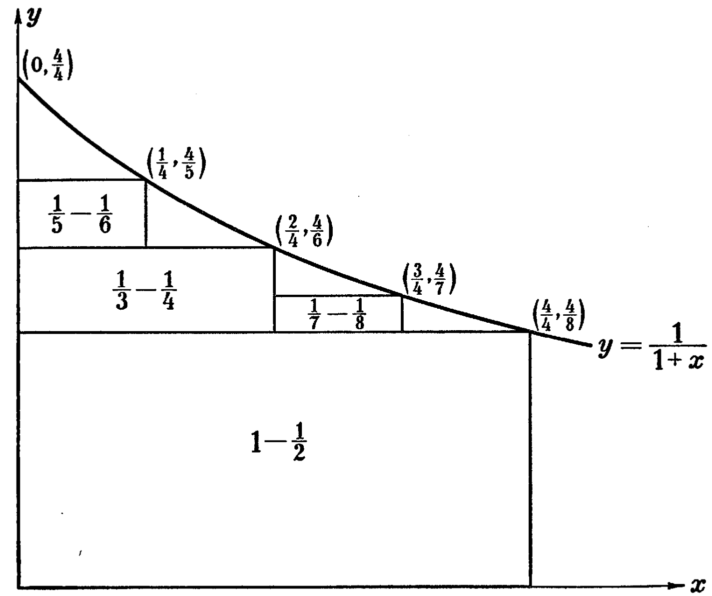

Fig. 1

* Lecture in the Norwegian Mathematical Society, Oct. 21, 1969.

bola $y=1/(1+x)$, when x goes from 0 to 1. Lord Brouncker used the same bisecting-summation as Archimedes did when he determined the area of the segment of a parabola. As seen from fig. 1, lord Brouncker obtained in this way the formula

$$A = \ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots .$$

In my article, I tried to divide in a corresponding manner the circular disc so that the formula of Leibniz

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots$$

could be "seen with the eyes", to use the words of Lionardo. I concluded my article in this way: "Obviously I have not succeeded in finding an equally "visible" formula for π as lord Brouncker found for $\ln 2$. It would surely be of great interest if someone could find a better "charting" of the area of the circle to illustrate this "arithmetical formula" for π, which certainly is one of the most glorious conquests in mathematics."

One of the reasons for not succeeding was that I did not know a simple deduction of the area under the curve $y=x^n$ when x goes from 0 to 1. A closer study of Fermat's method to calculate this area recently gave me the idea to use a similar procedure for the circular disc. Fermat [3] used a section of the interval from 0 to 1 following the terms of a geometric series, with quotient <1.

Let us circumscribe a circle quadrant with radius 1 by a square and divide the right squareside by the points with ordinates $1, k, k^2, \dots,$

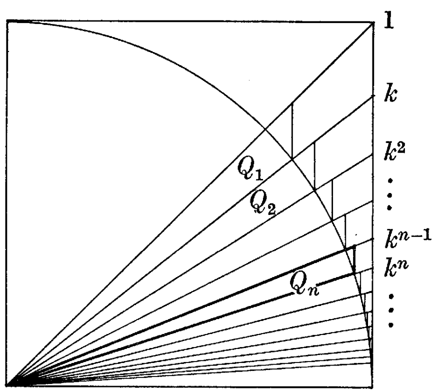

Fig. 2

where $k<1$ (see fig. 2). The calculation of the areas Q_n in the eighth part of the circle gives

$$Q_n = \frac{1-k}{2}\cdot\frac{k^{n-1}}{1+k^{2n}} = \frac{1-k}{2}k^{n-1}(1-k^{2n}+k^{4n}-k^{6n}+\dots),$$

valid for $k<1$ and $n \geqq 1$, and consequently

$$Q_1 = \frac{1-k}{2}(1-k^2+k^4-k^6+\dots)$$

$$Q_2 = \frac{1-k}{2}(k-k^5+k^9-k^{13}+\dots)$$

$$Q_3 = \frac{1-k}{2}(k^2-k^8+k^{14}-k^{20}+\dots)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

A vertical summation gives

$$\sum_{n=1}^{\infty} Q_n = \frac{1-k}{2}\left(\frac{1}{1-k}-\frac{k^2}{1-k^3}+\frac{k^4}{1-k^5}-\frac{k^6}{1-k^7}+\dots\right)$$

$$= \frac{1}{2}\left(1-\frac{k^2}{1+k+k^2}+\frac{k^4}{1+k+\dots+k^4}-\frac{k^6}{1+k+\dots+k^6}+\dots\right).$$

Let us also inscribe a "circular saw" in the eighth part of the circle (see fig. 3). A calculation of the areas P_n gives

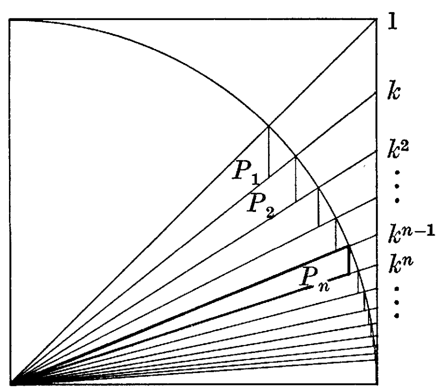

Fig. 3

$$P_n = \frac{1-k}{2}\cdot\frac{k^{n-1}}{1+k^{2n-2}} = \frac{1-k}{2}\,k^{n-1}(1-k^{2n-2}+k^{4n-4}-k^{6n-6}+\ldots)\,,$$

valid for $k<1$ and $n \geqq 2$. For $n=1$ we have

$$P_1 = \frac{1-k}{2}\cdot\frac{1}{2}\,.$$

A similar vertical summation as above gives

$$\sum_{n=1}^{\infty} P_n = \frac{1-k}{4}+\frac{1}{2}k\left(1-\frac{k^2}{1+k+k^2}+\frac{k^4}{1+k+\ldots+k^4}-\ldots\right).$$

Clearly

$$2\sum_{n=1}^{\infty} P_n < \frac{\pi}{4} < 2\sum_{n=1}^{\infty} Q_n\,,$$

where the difference between the upper and the lower bound is

$$(1-k)\left(\frac{1}{2}-\frac{k^2}{1+k+k^2}+\frac{k^4}{1+k+\ldots+k^4}-\ldots\right) < \frac{1-k}{2}.$$

Using the upper bound, we conclude that

$$\frac{\pi}{4} = \lim_{k\to 1}\left(1-\frac{k^2}{1+k+k^2}+\frac{k^4}{1+k+\ldots+k^4}-\frac{k^6}{1+k+\ldots+k^6}+\ldots\right),$$

and therefore

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\,.$$

If we compare with lord Brouncker's "mapping" of his hyperbolian area, we must admit that our procedure has not led to a division of our area in parts $1-\frac{1}{3}$, $\frac{1}{5}-\frac{1}{7}$, etc. It was necessary to apply a calculation with a limiting process.

Let us try to give a geometrical meaning to the terms

$$A_1 = \frac{1-k}{2}(1-k^2),\quad A_2 = \frac{1-k}{2}(k-k^5),\ \text{etc.}$$

in our vertical summation of Q_n, which are producing the term $1-\frac{1}{3}$ in $\pi/4$. It is possible to interpret $A_1, A_2, \ldots$ as trapezoids inside Q_1, $Q_2, \ldots$.

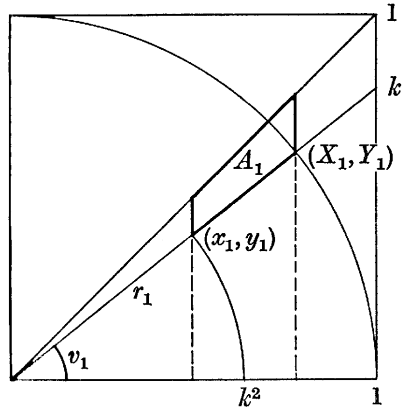

Fig. 4

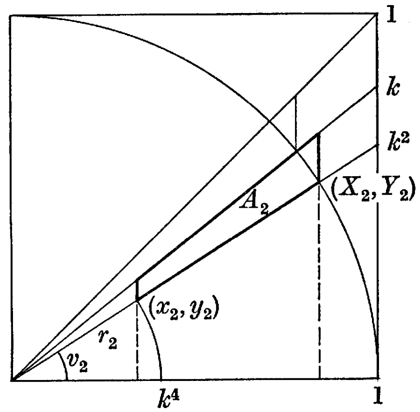

Fig. 5

In figs. 4 and 5, we have drawn circular arcs with radii k^2 and k^4 respectively. For the next trapezoid with area A_3, we must draw a circular arc with radius k^6, etc. From fig. 4 we see that

$$A_1 = \frac{1-k}{2}(X_1^2-x_1^2) = \frac{1-k}{2}\cdot\frac{1-k^4}{1+k^2} = \frac{1-k}{2}(1-k^2)\,,$$

as wanted. We also see that

$$x_1^2+y_1^2 = k^4 = \left(\frac{y_1}{x_1}\right)^4,\quad \text{or}\quad r_1 = \operatorname{tg}^2 v_1\,.$$

Likewise we get from fig. 5 that

$$A_2 = \frac{1-k}{2}(k-k^5)\,,$$

and

$$x_2^2+y_2^2 = k^8 = \left(\frac{y_2}{x_2}\right)^4,\quad \text{or}\quad r_2 = \operatorname{tg}^2 v_2\,.$$

Generally we get

$$x_n^2+y_n^2 = \left(\frac{y_n}{x_n}\right)^4,\quad \text{or}\quad r_n = \operatorname{tg}^2 v_n\,.$$

These equations are independent of k. We can therefore interprete geometrically the formula

$$\sum_{n=1}^{\infty} A_n = \frac{1}{2}\left(1-\frac{k^2}{1+k+k^2}\right)$$

as a sum of trapezoids, all of them having their lower left corner on the curve (fig. 6)

$$r = \operatorname{tg}^2 v .$$

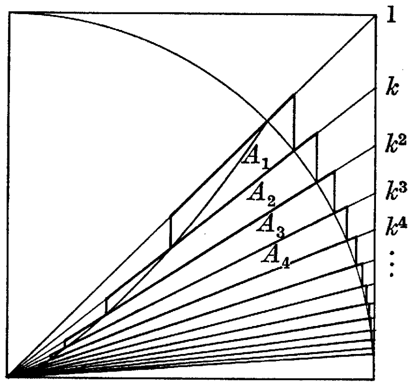

Fig. 6

Let us also study the terms giving rise to $\frac{1}{5}-\frac{1}{7}$ in $\pi/4$:

$$B_1 = \frac{1-k}{2}(1-k^2)k^4, \quad B_2 = \frac{1-k}{2}(1-k^4)k^9, \text{ etc.}$$

We can give these terms in our vertical summation a geometrical interpretation as trapezoids situated between A_1 and the origin, between A_2

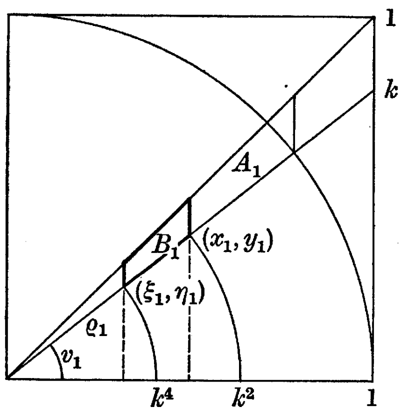

Fig. 7

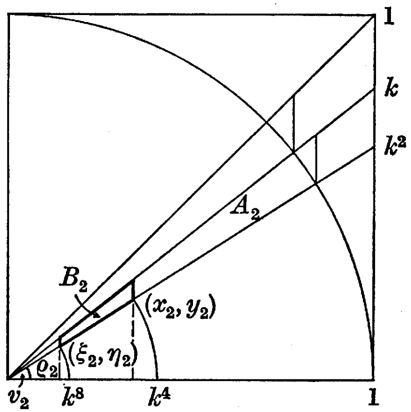

Fig. 8

and the origin, etc. A study of figs. 7 and 8 shows that the trapezoids B_1 and B_2 have the areas we want. Further

$$\xi_1^2+\eta_1^2 = k^8 = \left(\frac{\eta_1}{\xi_1}\right)^8, \quad \varrho_1 = \operatorname{tg}^4 v_1$$

$$\xi_2^2+\eta_2^2 = k^{16} = \left(\frac{\eta_2}{\xi_2}\right)^8, \quad \varrho_2 = \operatorname{tg}^4 v_2 .$$

In general we can deduce that the lower left corner of B_n is situated on the curve

$$r = \operatorname{tg}^4 v .$$

This leads to the conjecture that it is possible to construct a "map" of the eighthpart of the circular disc by means of the boundary curves

$$r = \operatorname{tg}^2 v, \quad r = \operatorname{tg}^4 v, \quad r = \operatorname{tg}^6 v, \ \ldots$$

or in cartesian coordinates:

$$x^6+x^4y^2 = y^4, \quad x^{10}+x^8y^2 = y^8, \quad x^{14}+x^{12}y^2 = y^{12}, \text{ etc.}$$

between parts with areas

$$\frac{1}{2}\left(1-\frac{1}{3}\right), \ \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right), \ \frac{1}{2}\left(\frac{1}{9}-\frac{1}{11}\right), \ \ldots .$$

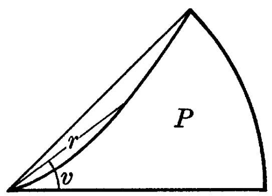

Fig. 9

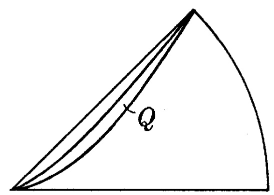

Fig. 10

It is necessary to verify this conjecture. (See figs. 9 and 10.) For the area P between the axis and the curves $r=1$ and $r=\operatorname{tg}^2 v$ we obtain

$$P = \frac{1}{2}\int_0^{\frac{1}{4}\pi} dv - \frac{1}{2}\int_0^{\frac{1}{4}\pi} \operatorname{tg}^4 v \, dv = \frac{1}{2}(I_0-I_4),$$

where we have introduced the notation

$$I_n = \int_0^{\frac{1}{4}\pi} \mathrm{tg}^n v\, dv\,.$$

For the area Q between the curves $r=\mathrm{tg}^2 v$ and $r=\mathrm{tg}^4 v$ we obtain

$$Q = \frac{1}{2}\int_0^{\frac{1}{4}\pi} \mathrm{tg}^4 v\, dv - \frac{1}{2}\int_0^{\frac{1}{4}\pi} \mathrm{tg}^8 v\, dv = \frac{1}{2}(I_4 - I_8), \text{ etc.}$$

It is obvious that $I_{n+1} < I_n$ because $\mathrm{tg}^{n+1} v < \mathrm{tg}^n v$ in the interval from 0 to $\pi/4$. Since

$$\int_0^{\frac{1}{4}\pi} \mathrm{tg}^{n-1} v(1+\mathrm{tg}^2 v)\, dv = \frac{1}{n}\Big|_0^{\frac{1}{4}\pi} \mathrm{tg}^n v = \frac{1}{n}\,,$$

we obtain the recursive formula

$$I_{n-1} + I_{n+1} = \frac{1}{n}\,.$$

From this and $I_n > 0$ we conclude that

$$\lim_{n\to\infty} I_n = 0\,.$$

Further

$$I_{n-1} - I_{n+3} = \frac{1}{n} - \frac{1}{n+2}$$

and therefore

$$I_0 - I_4 = 1 - \frac{1}{3},\ I_4 - I_8 = \frac{1}{5} - \frac{1}{7},\ I_8 - I_{12} = \frac{1}{9} - \frac{1}{11}, \ldots,$$

which verifies our conjecture.

We are now able to draw a "map" of the circular disc (see fig. 11). Here the four fan-shaped parts together have the area $4(1-\frac{1}{3})$, the eight greatest sicle-formed parts the area $4(\frac{1}{5}-\frac{1}{7})$, the next-greatest sicle-formed parts the area $4(\frac{1}{9}-\frac{1}{11})$, and so on. The equations for one set of limiting curves, in a suitable polar coordinate system, have the very simple form $r=\mathrm{tg}^{2n} v$, $n=1,2,3,\ldots$.

I suppose that Lionardo da Vinci would have been glad to see this figure "with his eyes"!

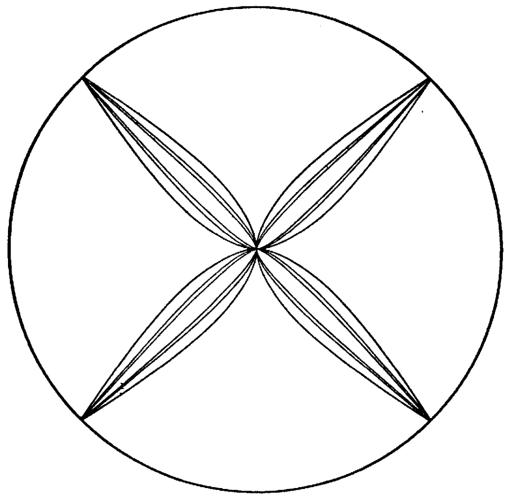

Fig. 11

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